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A New Combination Method for Solving Nonlinear Liouville-Caputo and Caputo-Fabrizio Time-Fractional Reaction-Diffusion-Convection Equations

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Abstract

In this article, we propose a new combination method called Shehu adomain decomposition method (SADM) to solve nonlinear time-fractional reaction-diffusion-convection equations with Liouville-Caputo and Caputo-Fabrizio fractional derivatives. This method is based on a combination of two powerful methods: the Shehu transform method and the Adomian decomposition method. The advantage of this method is that it is efficient, precise, and easy to implement with less computational effort. Applicability and theoretical results will be demonstrated and enhanced using a numerical example. Numerical results coupled with tables and graphical representations indicate that the proposed method is fully compatible with the complexity of these fractional equations and convenient to handle a various range of other fractional partial differential equations.

Keywords: Nonlinear time-fractional reaction-diffusion-convection equation; Liouville-Caputo fractional derivative; Caputo-Fabrizio fractional derivative; Shehu transform; Adomian decomposition method.

1 Introduction

Recently, fractional partial differential equations obtained from classical partial differential equations by replacing the ordinary order derivative with a fractional order derivative, have been treated by many mathematicians and physicists due to its multiple applications in various scientific fields such as: fluid mechanics [16], viscoelasticity [22], chemistry [24], diffusion [18], economics [4], biology [23], geophysics [26], bioengineering [19], and other areas of science. In all these scientific fields, it is important to obtain analytical or numerical solutions of fractional partial differential equations.

In the past few years, several combination methods have been proposed and developed to facilitate and improve the resolution speed of fractional partial differential equations. Among which: Laplace homotopy perturbation method [11], Laplace decomposition method [3], Local fractional Laplace variational iteration method [27], homotopy perturbation Sumudu transform method [28], homotopy analysis Sumudu transform method [15], variational iteration Sumudu transform method [1], fractional natural decomposition method [25], natural homotopy perturbation method [20], fractional Yang variational iteration method [2].

The objective of this article is to combine two powerful methods, the Shehu transform method and Adomian decomposition method to obtain a better method for solving a certain class of nonlinear fractional partial differential equations with two different fractional derivative operators. The advantage of this method lies in its ability to solve nonlinear fractional differential equations without resorting to linearization, perturbation, or discretization. In addition to this, its rapid convergence to the exact solution and so its high degree of accuracy.

Let the nonlinear Liouville-Caputo time-fractional reaction-diffusion-convection equation

$$D_t^{\alpha} u = (a(u)u_x)_x + b(u)u_x + c(u), \tag{1}$$

with the initial condition

$$u(x,0) = u_0(x), \tag{2}$$

where D_t^{α} is the Liouville-Caputo time-fractional derivative operator of order α, $0 < \alpha \leq 1$.

Let the nonlinear Caputo-Fabrizio time-fractional reaction-diffusion-convection equation

$$\mathcal{D}_t^{(\alpha)} u = (a(u)u_x)_x + b(u)u_x + c(u), \tag{3}$$

with the initial condition

$$u(x,0) = u_0(x), \tag{4}$$

where $\mathcal{D}_t^{(\alpha)}$ is the Caputo-Fabrizio time-fractional derivative operator of order α, $0 < \alpha \leq 1$.

In Equations (1) and (3), $u = \{u(x,t), x \in \mathbb{R}, t \geq 0\}$ is an unknown function and the arbitrary smooth functions $a(u), b(u)$ and $c(u)$ denote the diffusion term, the convection term and the reaction term respectively. The reaction-diffusion-convection problems are a very useful mathematical model in applied sciences such as biology [10], physics [5], chemistry [9], astrophysics [8], medicine and engineering [7].

The article is organized as follows. In Section 2, we give some preliminaries and definitions of the theory of fractional calculus. In Section 3, the Shehu transform and some of its essential properties are discussed. Section 4, is devoted to the analysis of Shehu adomian decomposition method

(SADM) for solving the nonlinear Liouville-Caputo and Caputo-Fabrizio time-fractional reaction-diffusion-convection equations (1) and (3) and present a numerical example for demonstrate the accuracy and effectiveness of the proposed method. In Section 5, we discuss our obtained results represented by figures and tables. Section 6 is for our conclusion of this study.

2 Preliminaries of Fractional Calculus

During the last few decades, mathematical concepts of fractional integrals and derivatives have been proposed according to several approaches such that Riemann-Liouville, Liouville-Caputo, Caputo-Fabrizio, Atangana-Baleanu and Conformable fractional derivative concepts. In this study, we utilize the Liouville-Caputo fractional derivative, which is a modification of Riemann-Liouville, because the initial conditions defined during the formulation of the problem are similar to those conventional conditions of integer order and the Caputo-Fabrizio fractional derivative, which is a modification of Liouville-Caputo fractional derivative. This fractional derivative was obtained by substituting the kernel in the Liouville-Caputo fractional derivative with an exponential function to get the fractional derivative without singular kernel.

Definition 2.1. [14] *Let $f \in L^1(0,T), T > 0$. The Riemann-Liouville fractional integral of order $\alpha \geq 0$ is defined by*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi, \tag{5}$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2.2. [14] *Let $f^{(n)} \in L^1(0,T), T > 0$. The Liouville-Caputo fractional derivative of order $\alpha \geq 0$ is defined by*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi, \tag{6}$$

where $n-1 < \alpha \leq n$, $n = [\alpha]+1$ with $[\alpha]$ being the integer part of α.

Definition 2.3. [14] *The Mittag-Leffler function is defined as follows:*

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}, \alpha \in \mathbb{C}, Re(\alpha) > 0, \tag{7}$$

for $\alpha = 1$, $E_{\alpha}(z)$ reduces to e^z.

In Equation (6) if transformations happen as follows:

$$(t-\xi)^{n-\alpha-1} \longrightarrow \exp\left[-\frac{\alpha(t-\xi)}{1-\alpha}\right] \text{ and } \frac{1}{\Gamma(n-\alpha)} \longrightarrow \frac{M(\alpha)}{1-\alpha}.$$

The new definition of fractional operator is expressed by Caputo and Fabrizio.

Definition 2.4. [6] *Let $f \in H^1(0,T), T > 0$. Then the Caputo-Fabrizio fractional derivative of order*

$\alpha, 0 < \alpha \leq 1$ *is defined as*

$$\begin{aligned} \mathcal{D}^{(\alpha)}f(t) &= \frac{M(\alpha)}{1-\alpha}\int_0^t f'(\xi)\exp\left[-\frac{\alpha\,(t-\xi)}{1-\alpha}\right]d\xi \\ &= \frac{M(\alpha)}{1-\alpha}\left(f'(t) * \exp\left[-\frac{\alpha t}{1-\alpha}\right]\right), \end{aligned} \tag{8}$$

where $*$ *denotes the convolution and* $M(\alpha)$ *is a normalization function that satisfies* $M(0) = M(1) = 1$.

From Equation (8) it follows that if $f(t) = C$ is a constant, then $\mathcal{D}^{(\alpha)}C = 0$ as in the sense of Caputo [14].

If $f \notin H^1(0,T)$, then its fractional derivative is redefined as in [6],

$$\mathcal{D}^{(\alpha)}f(t) = \frac{\alpha M(\alpha)}{1-\alpha}\int_0^t (f(t)-f(\xi))\exp\left[-\frac{\alpha\,(t-\xi)}{1-\alpha}\right]d\xi, t > 0.$$

For $n \geq 1$ and $0 < \alpha \leq 1$, the fractional derivative of order $(\alpha + n)$ is defined by

$$\mathcal{D}^{(\alpha+n)}f(t) = \mathcal{D}^{(\alpha)}(\mathcal{D}^{(n)}f(t)). \tag{9}$$

The above Caputo-Fabrizio fractional derivative was later modified by Jorge Losada and Juan José Nieto [17] as

$$\mathcal{D}^{(\alpha)}f(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}\int_0^t f'(\tau)\exp\left[-\frac{\alpha\,(t-\xi)}{1-\alpha}\right]d\xi, t > 0. \tag{10}$$

The fractional integral corresponding to the derivative in Equation (10) was defined by Jorge Losada and Juan José Nieto in 2015, as follows.

Definition 2.5. [17] *Let* $0 < \alpha \leq 1$. *The fractional integral of order* α *of* f *is defined by*

$$\mathcal{I}^{(\alpha)}f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}f(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t f(\xi)d\xi, t > 0. \tag{11}$$

From the definition in Equation (11), the fractional integral of Caputo-Fabrizio type of a function f of order $0 < \alpha \leq 1$ is an average between function f and its one order integral, i.e.

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1.$$

Therefore,

$$M(\alpha) = \frac{2}{2-\alpha}, 0 < \alpha \leq 1.$$

Due to this, Losada and Nieto remarked that Caputo-Fabrizio fractional derivative can be redefined as follows:

Definition 2.6. [17] *Let $0 < \alpha \leq 1$. The Caputo-Fabrizio fractional derivative of order α of a function f is given by*

$$\mathcal{D}^{(\alpha)} f(t) = \frac{1}{1-\alpha} \int_0^t f'(\xi) \exp\left[-\frac{\alpha(t-\xi)}{1-\alpha}\right] d\xi, t > 0, \tag{12}$$

and its fractional integral is defined as

$$\mathcal{I}^{(\alpha)} f(t) = (1-\alpha) f(t) + \alpha \int_0^t f(\xi) d\xi, t > 0.$$

3 The Shehu Transform

The Shehu transform, introduced by [21], is an exponential type kernel integral transform which also generalizes the Laplace and Sumudu integral transforms.

Definition 3.1. [21] *The Shehu transform of the function $f(t)$ of exponential order is defined over the set of functions*

$$A = \left\{ f(t)/\exists N, \eta_1, \eta_2 > 0, |f(t)| < N \exp\left(\frac{|t|}{\eta_j}\right), \text{ if } t \in (-1)^i \times [0, \infty) \right\},$$

by the following integral

$$\mathbb{S}[f(t)] = F(s,v) = \int_0^\infty \exp\left(-\frac{st}{v}\right) f(t) dt, t > 0.$$

Some basic properties of the Shehu transform are given as follows:

Property 1: The Shehu transform is a linear operator. That is, if λ and μ are non-zero constants, then

$$\mathbb{S}[\lambda f(t) \pm \mu g(t)] = \lambda \mathbb{S}[f(t)] \pm \mu \mathbb{S}[g(t)].$$

Property 2: If $f^{(n)}(t)$ is the $n-$th derivative of the function $f(t) \in A$ with respect to "t" then its Shehu transform is given by

$$\mathbb{S}\left[f^{(n)}(t)\right] = \frac{s^n}{v^n} F(s,v) - \sum_{k=0}^{n-1} \left(\frac{s}{v}\right)^{n-(k+1)} f^{(k)}(0).$$

Property 3: (Convolution property) Suppose $F(s,v)$ and $G(s,v)$ are the Shehu transforms of $f(t)$ and $g(t)$, respectively, both defined in the set A. Then the Shehu transform of their convolution is given by

$$\mathbb{S}[(f * g)(t)] = F(s,v) G(s,v),$$

where the convolution of two functions is defined by

$$(f * g)(t) = \int_0^t f(\xi) g(t-\xi) d\xi = \int_0^t f(t-\xi) g(\xi) d\xi.$$

Property 4: Some special Shehu transforms

$$\begin{aligned}\mathbb{S}(1) &= \frac{v}{s},\\ \mathbb{S}(t) &= \frac{v^2}{s^2},\\ \mathbb{S}\left(\frac{t^n}{n!}\right) &= \left(\frac{v}{s}\right)^{n+1}, n = 0, 1, 2, \ldots .\end{aligned}$$

Property 5: The Shehu transform of t^α is given by

$$\mathbb{S}\left[t^\alpha\right] = \left(\frac{v}{s}\right)^{\alpha+1} \Gamma\left(\alpha+1\right).$$

Theorem 3.1. [12] *Let $n \in \mathbb{N}^*$ and $\alpha > 0$ be such that $n-1 < \alpha \leq n$ and $F(s, v)$ be the Shehu transform of the function $f(t)$, then the Shehu transform denoted by $F_\alpha(s, v)$ of the Liouville-Caputo fractional derivative of $f(t)$ of order α, is given by*

$$\mathbb{S}\left[D^\alpha f(t)\right] = F_\alpha(s, v) = \frac{s^\alpha}{v^\alpha} F(s, v) - \sum_{k=0}^{n-1} \left(\frac{s}{v}\right)^{\alpha-(k+1)} \left[D^k f(t)\right]_{t=0}. \tag{13}$$

Proof. See. [12]. Theorem 3.7. □

Theorem 3.2. *The Shehu transform of the Caputo-Fabrizio fractional derivative of the function $f(t)$ of order $(\alpha + n)$ where $0 < \alpha \leq 1$ and $n \in \mathbb{N} \cup 0$, is given by*

$$\mathbb{S}\left[\mathcal{D}^{(\alpha+n)} f(t)\right] = \frac{v}{s - \alpha(s-v)} \left[\left(\frac{s}{v}\right)^{n+1} \mathbb{S}\left(f(t)\right) - \sum_{k=0}^{n} \left(\frac{s}{v}\right)^{n-k} f^{(k)}(0)\right]. \tag{14}$$

Proof. By the definition of the Caputo-Fabrizio fractional derivative 2.6 and the relation (9), we have

$$\begin{aligned}\mathbb{S}\left[\mathcal{D}^{(n+\alpha)} f(t)\right] &= \mathbb{S}\left[\mathcal{D}^{(\alpha)}(\mathcal{D}^{(n)} f(t))\right]\\ &= \frac{1}{1-\alpha} \int_0^{+\infty} \exp\left(-\frac{st}{v}\right) \int_0^t f^{(n)}(\xi) \exp\left[-\frac{\alpha\left(t-\xi\right)}{1-\alpha}\right] d\xi\\ &= \frac{1}{1-\alpha} \int_0^{+\infty} \exp\left(-\frac{st}{v}\right) \left(f^{(n)}(t) * \exp\left[-\frac{\alpha t}{1-\alpha}\right]\right)\\ &= \frac{1}{1-\alpha} \mathbb{S}\left(f^{(n)}(t) * \exp\left[-\frac{\alpha t}{1-\alpha}\right]\right).\end{aligned}$$

Hence, from the properties of the Shehu transform, we have

$$\begin{aligned}\mathbb{S}\left[\mathcal{D}^{(\alpha+n)} f(t)\right] &= \frac{1}{1-\alpha} \mathbb{S}\left(f^{(n)}(t)\right) \mathbb{S}\left(\exp\left[-\frac{\alpha t}{1-\alpha}\right]\right)\\ &= \frac{v}{s-\alpha(s-v)} \left[\frac{s^n}{v^n} \mathbb{S}\left(f(t)\right) - \sum_{k=0}^{n-1} \left(\frac{s}{v}\right)^{n-(k+1)} f^{(k)}(0)\right]\\ &= \frac{v}{s-\alpha(s-v)} \left[\left(\frac{s}{v}\right)^{n+1} \mathbb{S}\left(f(t)\right) - \sum_{k=0}^{n} \left(\frac{s}{v}\right)^{n-k} f^{(k)}(0)\right].\end{aligned}$$

The proof is complete. □

4 Analysis of the Shehu Adomian Decomposition Method (SADM)

The purpose of this section is to present the methodology of the Shehu adomian decomposition method (SADM) to solve the nonlinear Liouville-Caputo and Caputo-Fabrizio time-fractional reaction-diffusion-convection equations and demonstrate it by solving a numerical example.

Theorem 4.1. *Let us consider the following nonlinear Liouville-Caputo and Caputo-Fabrizio time-fractional reaction-diffusion-convection equations (1) and (3). Then by SADM, the solution of Equations (1) and (3) is given in the form of an infinite series which converges rapidly to the exact solution as follows:*

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{15}$$

Proof. To prove the above Theorem, we define

$$\begin{aligned} \mathcal{N}u &= (a(u)u_x)_x\,, \\ \mathcal{M}u &= b(u)u_x, \\ \mathcal{K}u &= c(u). \end{aligned}$$

Firstly, we consider the following nonlinear Liouville-Caputo time-fractional reaction-diffusion-convection equation (1) with the initial condition (2).

Equation (1) is written in the form

$$D_t^{\alpha} u = \mathcal{N}u + \mathcal{M}u + \mathcal{K}u. \tag{16}$$

Taking the Shehu transform on two sides of Equation (16) and using the Theorem 3.1, to get

$$\mathbb{S}\left[u\right] = \frac{v}{s} u_0(x) + \frac{v^{\alpha}}{s^{\alpha}} \mathbb{S}\left[\mathcal{N}u + \mathcal{M}u + \mathcal{K}u\right]. \tag{17}$$

Then, we apply the inverse Shehu transform on two sides of Equation (17), we have

$$u = u_0(x) + \mathbb{S}^{-1}\left(\frac{v^{\alpha}}{s^{\alpha}} \mathbb{S}\left[\mathcal{N}u + \mathcal{M}u + \mathcal{K}u\right]\right). \tag{18}$$

Now, we represent the solution in an infinite series form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{19}$$

and the nonlinear terms can be decomposed as follows:

$$\begin{aligned}\mathcal{N}u &= \sum_{n=0}^{\infty} A_n, \\ \mathcal{M}u &= \sum_{n=0}^{\infty} B_n, \\ \mathcal{K}u &= \sum_{n=0}^{\infty} C_n,\end{aligned} \tag{20}$$

where A_n, B_n and C_n are Adomian polynomials [29], of $u_0, u_1, ..., u_n$ and it can be calculated by the formula given below:

$$A_n = B_n = C_n = \frac{1}{n!}\frac{\partial^n}{\partial \lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, n = 0, 1, 2, \ldots.$$

Using Equations (19) and (20), we can rewrite Equation (18) as

$$\sum_{n=0}^{\infty} u_n(x,t) = u_0(x) + \mathbb{S}^{-1}\left(\frac{v^\alpha}{s^\alpha}\mathbb{S}\left[\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n + \sum_{n=0}^{\infty} C_n\right]\right). \tag{21}$$

By comparing both sides of Equation (21), we have the following relation:

$$\begin{aligned}u_0(x,t) &= u_0(x), \\ u_1(x,t) &= \mathbb{S}^{-1}\left(\frac{v^\alpha}{s^\alpha}\mathbb{S}\left[A_0 + B_0 + C_0\right]\right), \\ u_2(x,t) &= \mathbb{S}^{-1}\left(\frac{v^\alpha}{s^\alpha}\mathbb{S}\left[A_1 + B_1 + C_1\right]\right), \\ u_3(x,t) &= \mathbb{S}^{-1}\left(\frac{v^\alpha}{s^\alpha}\mathbb{S}\left[A_2 + B_2 + C_2\right]\right), \\ &\vdots\end{aligned}$$

and so on.

In general, the recursive relation is given by

$$\begin{aligned}u_0(x,t) &= u_0(x), \\ u_{n+1}(x,t) &= \mathbb{S}^{-1}\left(\frac{v^\alpha}{s^\alpha}\mathbb{S}\left[A_n + B_n + C_n\right]\right), n \geq 0.\end{aligned} \tag{22}$$

Finally, the solution of the Equation (16) is given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

Secondly, we consider the following nonlinear Caputo-Fabrizio time-fractional reaction-diffusion-convection equation (3) with the initial condition (4).

Equation (3) is written in the form

$$\mathcal{D}_t^{(\alpha)} u = \mathcal{N}u + \mathcal{M}u + \mathcal{K}u. \tag{23}$$

Taking the Shehu transform on two sides of Equation (23) and using the Theorem 3.2, to get

$$\mathbb{S}\left[u\right] = \frac{v}{s}u_0(x) + \frac{s-\alpha(s-v)}{s}\mathbb{S}\left[\mathcal{N}u + \mathcal{M}u + \mathcal{K}u\right]. \tag{24}$$

Then, we apply the inverse Shehu transform on two sides of Equation (24), we have

$$u = u_0(x) + \mathbb{S}^{-1}\left(\frac{s-\alpha(s-v)}{s}\mathbb{S}\left[\mathcal{N}u + \mathcal{M}u + \mathcal{K}u\right]\right). \tag{25}$$

Now, we represent the solution in an infinite series form (19) and the nonlinear terms can be decomposed in the form (20)

Thus, Equation (25) can rewrite as follows:

$$\sum_{n=0}^{\infty} u_n(x,t) = u_0(x) + \mathbb{S}^{-1}\left(\frac{s-\alpha(s-v)}{s}\mathbb{S}\left[\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n + \sum_{n=0}^{\infty} C_n\right]\right). \tag{26}$$

By comparing both sides of Equation (26), we have the following relation:

$$\begin{aligned}
u_0(x,t) &= u_0(x),\\
u_1(x,t) &= \mathbb{S}^{-1}\left(\frac{s-\alpha(s-v)}{s}\mathbb{S}\left[A_0 + B_0 + C_0\right]\right),\\
u_2(x,t) &= \mathbb{S}^{-1}\left(\frac{s-\alpha(s-v)}{s}\mathbb{S}\left[A_1 + B_1 + C_1\right]\right),\\
u_3(x,t) &= \mathbb{S}^{-1}\left(\frac{s-\alpha(s-v)}{s}\mathbb{S}\left[A_2 + B_2 + C_2\right]\right),\\
&\vdots
\end{aligned}$$

and so on.

In general, the recursive relation is given by

$$\begin{aligned}
u_0(x,t) &= u_0(x),\\
u_{n+1}(x,t) &= \mathbb{S}^{-1}\left(\frac{s-\alpha(s-v)}{s}\mathbb{S}\left[A_n + B_n + C_n\right]\right), \; n \geq 0.
\end{aligned} \tag{27}$$

Finally, the solution of the Equation (23) is given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t).$$

The proof is complete. □

Theorem 4.2. *Let $\mathcal{B}$ be a Banach space, then the series solutions of the Equations. (1) and (3) converges to $S \in \mathcal{B}$, if there exists γ, $0 < \gamma < 1$ such that*

$$\|u_n\| \leq \gamma \|u_{n-1}\|, \forall n \in \mathbb{N}.$$

Proof. Define the sequence $\{S_n\}_{n\geq 0}$ of partial sums of the series given by the recursive relation (22) or (27) as follows:

$$S_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + ... + u_n(x,t),$$

and we need to show that $\{S_n\}_{n\geq 0}$ is a Cauchy sequence in Banach space $\mathcal{B}$. For this purpose, we consider

$$\|S_{n+1} - S_n\| \leq \|u_{n+1}\| \leq \gamma \|u_n\| \leq \gamma^2 \|u_{n-1}\| \leq ... \leq \gamma^{n+1} \|u_0\|. \tag{28}$$

For every $n, m \in \mathbb{N}$, $n \geq m$, by using (28) and the triangle inequality successively, we have

$$\begin{aligned}
\|S_n - S_m\| &= \|S_{m+1} - S_m + S_{m+2} - S_{m+1} + ... + S_n - S_{n-1}\| \\
&\leq \|S_{m+1} - S_m\| + \|S_{m+2} - S_{m+1}\| + ... + \|S_n - S_{n-1}\| \\
&\leq \gamma^{m+1} \|u_0\| + \gamma^{m+2} \|u_0\| + ... + \gamma^n \|u_0\| \\
&= \gamma^{m+1} \left(1 + \gamma + ... + \gamma^{n-m-1}\right) \|u_0\| \\
&\leq \gamma^{m+1} \left(\frac{1 - \gamma^{n-m}}{1 - \gamma}\right) \|u_0\|.
\end{aligned}$$

Since $0 < \gamma < 1$, so $1 - \gamma^{n-m} \leq 1$, then

$$\|S_n - S_m\| \leq \frac{\gamma^{m+1}}{1 - \gamma} \|u_0\|.$$

Since u_0 is bounded, then

$$\lim_{n,m \longrightarrow \infty} \|S_n - S_m\| = 0.$$

Therefore, the sequence $\{S_n\}_{n\geq 0}$ is Cauchy sequence in the Banach space $\mathcal{B}$, so the series solution defined in (15) converges. This completes the proof. □

Theorem 4.3. *The maximum absolute truncation error of the series solution (15) of Equation (1) or (3) is estimated to be*

$$\sup_{(x,t)\in\Omega} \left| u_n(x,t) - \sum_{k=0}^{m} u_k(x,t) \right| \leq \frac{\gamma^{m+1}}{1 - \gamma} \sup_{(x,t)\in\Omega} |u_0(x,t)|, \tag{29}$$

where the region $\Omega \subset \mathbb{R} \times \mathbb{R}^+$.

Proof. From the Theorem 4.2, we have

$$\|S_n - S_m\| \leq \frac{\gamma^{m+1}}{1 - \gamma} \sup_{(x,t)\in\Omega} |u_0(x,t)|. \tag{30}$$

But we assume that $S_n = \sum_{k=0}^{n} u_k(x,t)$ and since $n \to \infty$, we obtain $S_n \to u_n(x,t)$, so (30) can be rewritten as

$$\|u_n(x,t) - S_m\| = \left\| u_n(x,t) - \sum_{k=0}^{m} u_k(x,t) \right\| \leq \frac{\gamma^{m+1}}{1-\gamma} \sup_{(x,t)\in\Omega} |u_0(x,t)| .$$

So, the maximum absolute truncation error in the region $\Omega \subset \mathbb{R} \times \mathbb{R}^+$ is

$$\sup_{(x,t)\in\Omega} \left| u_n(x,t) - \sum_{k=0}^{m} u_k(x,t) \right| \leq \frac{\gamma^{m+1}}{1-\gamma} \sup_{(x,t)\in\Omega} |u_0(x,t)| .$$

This completes the proof. □

Example: Consider the nonlinear Liouville-Caputo time-fractional reaction-diffusion-convection equation

$$D_t^\alpha u = u_{xx} + u u_x + u - u^2, \tag{31}$$

with the initial condition

$$u(x,0) = 1 + e^x, \tag{32}$$

where D_t^α is the Liouville-Caputo time-fractional derivative operator of order α, $0 < \alpha \leq 1$ and u is a function of $x, t \in \mathbb{R} \times \mathbb{R}^+$.

For $\alpha = 1$, the exact solution of Equations (31) and (32) is (see. [13])

$$u(x,t) = 1 + e^{x+t}.$$

Following the analysis of the SADM presented above, we have

$$\begin{aligned}
u_0(x,t) &= 1 + e^x, \\
u_1(x,t) &= \frac{t^\alpha}{\Gamma(\alpha+1)} e^x, \\
u_2(x,t) &= \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} e^x, \\
u_3(x,t) &= \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} e^x, \\
&\vdots
\end{aligned}$$

and so on.

So, the approximate solution of Equations (31) and (32) is given by

$$\begin{aligned}
u(x,t) &= 1 + e^x \left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots \right) \\
&= 1 + e^x \sum_{n=0}^{\infty} \frac{(t^\alpha)^n}{\Gamma(n\alpha+1)} = 1 + e^x E_\alpha(t^\alpha),
\end{aligned}$$

where $E_\alpha(t^\alpha)$ is the Mittag-Leffler function defined by Equation (7).

Therefore, the exact solution of Equations (31) and (32) when $\alpha \longrightarrow 1$, is given by

$$u(x,t) = 1 + e^{x+t}.$$

Now, we consider the nonlinear Caputo-Fabrizio time-fractional reaction-diffusion-convection equation

$$\mathcal{D}_t^{(\alpha)} u = u_{xx} + u u_x + u - u^2, \tag{33}$$

with the initial condition

$$u(x,0) = 1 + e^x, \tag{34}$$

where $\mathcal{D}_t^{(\alpha)}$ is the Caputo-Fabrizio fractional derivative operator of order α, $0 < \alpha \leq 1$ and u is a function of $x, t \in \mathbb{R} \times \mathbb{R}^+$.

For $\alpha = 1$, the exact solution of Equations (33) and (34) is (see. [13])

$$u(x,t) = 1 + e^{x+t}.$$

Following the analysis of the SADM presented above, we have

$$\begin{aligned}
u_0(x,t) &= 1 + e^x, \\
u_1(x,t) &= (1 - \alpha + \alpha t)\, e^x, \\
u_2(x,t) &= \left((1-\alpha)^2 + 2\alpha(1-\alpha)t + \alpha^2 \frac{t^2}{2!}\right) e^x, \\
u_3(x,t) &= \left((1-\alpha)^3 + 3\alpha(1-\alpha)^2 t + 3\alpha^2(1-\alpha)\frac{t^2}{2!} + \alpha^3 \frac{t^3}{3!}\right) e^x, \\
&\vdots
\end{aligned}$$

and so on.

Thus, the approximate solution of Equations (33) and (34) is given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots \tag{35}$$

Taking $\alpha = 1$ in (35), the solution of the Equations (33) and (34) has the general pattern form which is coinciding with the following exact solution in terms of infinite series,

$$u(x,t) = 1 + e^x \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right).$$

Hence, the exact solution of Equations (33) and (34) in a closed form of elementary function will be

$$u(x,t) = 1 + e^{x+t}.$$

5 Numerical Results and Discussion

In this section, the numerical results for Example 1 are presented. Figure 1 represent the surface graph of the $4-$term approximate solutions by SADM Liouville-Caputo at $\alpha = 0.8, 0, 9, 1$ and exact solution for Equation (31). Figure 2 represent the surface graph of the $4-$term approximate solutions by SADM Caputo-Fabrizio at $\alpha = 0.8, 0, 9, 1$ and exact solution for Equation (33). Figures 3 and 4 represent the behavior of the $4-$term approximate solutions at $t = 1$ and $x = 1$ where $\alpha = 0.7, 0.8, 0.95, 1$ and exact solution for Equations (31) and (33), respectively. According to these figures, we can say that when the order of the fractional derivative tends to 1, the approximate solutions obtained by SADM tends continuously to the exact solutions. Tables 1 and 2 show the numerical values of the $4-$term approximate solution and exact solution for different values of α and t at $x = 0.1$ for Equations (31) and (33), respectively. From the above, numerical results confirm that the approximate solutions are in good agreement with exact solutions at $\alpha = 1$ for Equations (31) and (33), respectively, and the high precision of the proposed method. Table 3 show the comparision of the $4-$term approximate solutions by generalized Taylor fractional series method (GTFSM) [13] and SADM at $\alpha = 0$ and $x = 0.1$. We conclude from this comparison that the solutions obtained by the proposed method are in complete agreement with the solutions available in the literature. Also it is to be noted that the accuracy can be improved by computing more terms of approximated solutions.

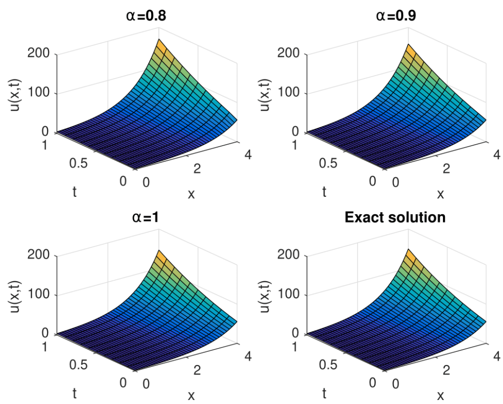

Figure 1: The surface graph of the $4-$term approximate solutions by SADM Liouville-Caputo and exact solution for Equation (31).

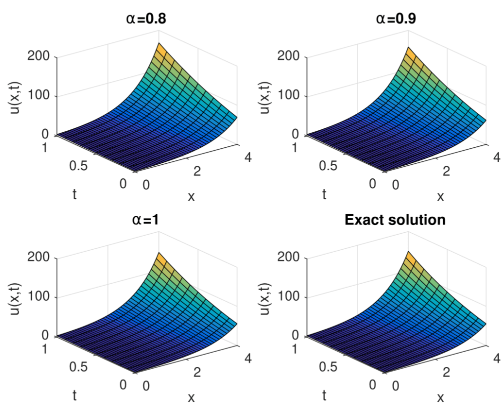

Figure 2: The surface graph of the $4-$term approximate solutions by SADM Caputo-Fabrizio and exact solution for Equation (33).

Table 1: The numerical values of the 4-term approximate solution by SADM Liouville-Caputo and exact solution of Equation (31) when $x = 0.1$.

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	u_{exact}	$\lvert u_{exact} - u_{SADM-LC} \rvert$
0.01	2.7231	2.6939	2.6761	2.6653	2.6653	6.8834×10^{-10}
0.03	2.8149	2.7601	2.7236	2.6989	2.6989	5.5980×10^{-8}
0.05	2.8930	2.8198	2.7690	2.7333	2.7333	4.3368×10^{-7}
0.07	2.9657	2.8769	2.8138	2.7683	2.7683	1.6728×10^{-6}
0.09	3.0354	2.9328	2.8585	2.8040	2.8040	4.5896×10^{-6}

Table 2: The numerical values of the 4-term approximate solution by SADM Caputo-Fabrizio and exact solution of Equation (33) when $x = 0.1$.

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	u_{exact}	$\lvert u_{exact} - u_{SADM-CF} \rvert$
0.01	3.3579	3.0777	2.8501	2.6653	2.6653	6.8834×10^{-10}
0.03	3.4017	3.1185	2.8873	2.6989	2.6989	5.5980×10^{-8}
0.05	3.4461	3.1600	2.9252	2.7333	2.7333	4.3368×10^{-7}
0.07	3.4911	3.2021	2.9638	2.7683	2.7683	1.6728×10^{-6}
0.09	3.5368	3.2450	3.0032	2.8040	2.8040	4.5896×10^{-6}

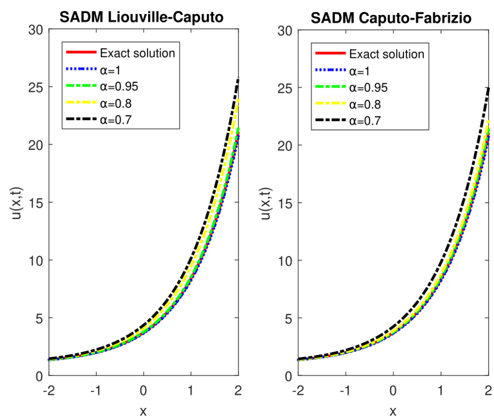

Figure 3: The behavior of the $4-$term approximate solution by SADM Liouville-Caputo and Caputo-Fabrizio for Equations (31) and (33) respectively, and exact solution, when $t = 1$.

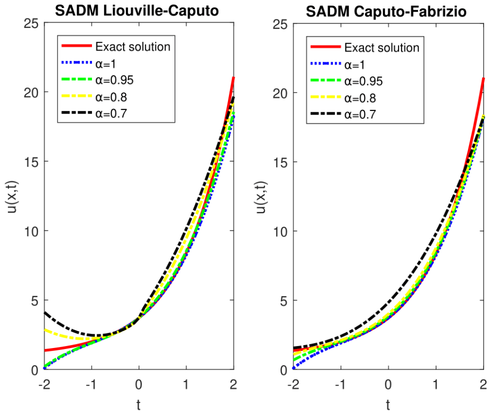

Figure 4: The behavior of the 4−term approximate solution by SADM Liouville-Caputo and Caputo-Fabrizio for Equations (31) and (33) respectively, and exact solution, when $x = 1$.

Table 3: The comparison of the 4-term approximate solutions by GTFSM, SADM Liouville-Caputo and SADM Caputo-Fabrizio when $\alpha = 1$ and $x = 0.1$.

t	GTFSM	SADM L-C	SADM C-F
0.01	2.6653	2.6653	2.6653
0.03	2.6989	2.6989	2.6989
0.05	2.7333	2.7333	2.7333
0.07	2.7683	2.7683	2.7683
0.09	2.8040	2.8040	2.8040

6 Conclusions

In this article, a new combination method called the Shehu adomian decomposition method (SADM) was successfully presented for solving the nonlinear Liouville-Caputo and Caputo-Fabrizio time-fractional reaction-diffusion-convection equations. A numerical example is presented to illustrate the effective and accurate of the proposed method. The obtained results are compared with the results of the existing methods. The SADM proved to be a powerful mathematical tool for solving nonlinear fractional partial differential equations and can further be extended to more complex fractional differential equations which arise in applied science and engineering.

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Conflicts of Interest The authors declare that there is no conflict of interest in this article.

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